Linear Dynamics & Composition Operators

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1. Linear Dynamics

Definition: X a TVS, $T : X \to X$ linear, contin.

"*T hypercyclic*" means "$\exists\, x \in X$ with *orbit* $\{T^n x\}_0^\infty$ dense."

Some history

1920's G. D. Birkhoff: *"Translation operator" $f(z) \to f(z+1)$ is hypercyclic on the space of entire functions*

1950's G. MacLane: *Same for differentiation*

1960's S. Rolewicz: $2B$ is hypercyclic on ℓ^2

1980's Carol Kitai: "Hypercyclicity Criterion," spectra, further examples.

1990's Various authors: More classes of examples

2000+ Younger generation: A beautiful theory develops ...

... Special Session on Linear Dynamics !!

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2. Composition operators on H^2

Setting

- $\mathbb{U} = \{|z| < 1\} \subset \mathbb{C}$
- H^2 = Hilbert space of f holomorphic in $\mathbb{U}$

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \qquad \text{with} \qquad \|f\|^2 := \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$$

Composition operators

- $\varphi : \mathbb{U} \to \mathbb{U}$ holomorphic
- $C_\varphi : H(\mathbb{U}) \to H(\mathbb{U})$ defined by: $C_\varphi f = f \circ \varphi$

Littlewood's Theorem (1920's):

$C_\varphi : H^2 \to H^2$ bounded operator

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3. Questions about $C_\varphi : H^2 \to H^2$

Compactness:

"Size of $C_\varphi(\text{Ball}\, H^2)$" vs. "Size of $\varphi(\mathbb{U})$"

Spectra: esp. eigenvalues/eigenfunctions

$f \circ \varphi = \lambda f$ (Schroeder's eqn, latter 1800's)

Dynamics / Cyclicity:

$$C_\varphi^n = C_{\varphi_n} \qquad (\varphi_n = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n \text{ times}})$$

Questions for today:

- Dynamics of φ on $\mathbb{U}$ vs. Hypercyclicity of C_φ

  P. Bourdon & JHS 1990's

- Dynamics of (inner) φ on $\partial\mathbb{U}$ vs. "operator-level-dynamics" of C_φ

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4. Hypercyclic C_φ: Example

$$\varphi(z) = \frac{1+3z}{3+z}$$

Properties

- "Conformal automorphism" of $\mathbb{U}$
- "Hyperbolic" (Two fixed pts, multiplier > 0):

  $+1$: The attractive fixed pt. ($\varphi'(1) = 1/2$)

  -1: The repulsive fixed pt. ($\varphi'(-1) = 2$)

Theorem

C_φ is hypercyclic on H^2

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5. Proof that $C_{\frac{1+3z}{3z+1}}$ is hypercyclic

Sufficient to satisfy hypotheses of *Hypercyclicity criterion**

$$\exists \textit{ dense } X, Y \subset H^2 \;\; \& \;\; \textit{map } S : Y \to Y \textit{ such that}$$

$$\|C_\varphi^n f\| \to 0 \quad \forall f \in X$$

$$C_\varphi S = I_Y$$

$$\|S^n g\| \to 0 \quad \forall g \in Y$$

The "Usual Suspects"

$$X = \{f \in \mathrm{Rat}(\mathbb{U}) : f(1) = 0\}.$$
$$S = C_\varphi^{-1} \quad (= C_{\varphi^{-1}})$$
$$Y = \{g \in \mathrm{Rat}(\mathbb{U}) : g(-1) = 0\}$$

Q.E.D

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6. Hypercyclic C_φ: $\varphi \in$ LFT($\mathbb{U}$)

Necessary for C_φ hypercyclic (any φ !!)

- φ has no fixed point in $\mathbb{U}$
- φ is 1-1 on $\mathbb{U}$

Theorem (w/ Paul Bourdon, 1990's)

For $\varphi \in LFT(\mathbb{U})$ with no fixed point in $\mathbb{U}$:

C_φ hypercyclic on H^2 unless φ is parabolic, non-automorphic.

$\varphi \in$ LFT($\mathbb{U}$) *parabolic* means

- $\varphi \approx \Phi : \text{RHP} \to \text{RHP}$
  $\Phi(w) = w + a, \ \ \exists\, a \neq 0 \ \text{ with } \ \operatorname{Re} a \geq 0$
- φ nonauto $\iff \operatorname{Re} a > 0 \iff \rho(\varphi_{n+1}(0), \varphi_n(0)) \to 0$

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7. Dynamics of φ on $\mathbb{U}$ (1920's)

The Denjoy-Wolff Theorem

φ fixes no point of $\mathbb{U}$

$\Longrightarrow$

$\exists\, \omega \in \partial\mathbb{U}$ *such that* $\varphi_n \to \omega$ *unif. on compacts*

The Julia-Carathéodory Theorem

ω = *"Denjoy-Wolff pt." of* $\varphi \in \partial\mathbb{U}$

$\Longrightarrow$

- $\displaystyle\lim_{z\to\omega\angle} \varphi(z) = \omega$
- $\displaystyle\lim_{z\to\omega\angle} \frac{\varphi(z)-\omega}{z-\omega} := \varphi'(\omega)$ *exists,* $\in (0,1]$

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8. Linear Fractional Models

Theorem ((Koenigs), Valiron, Baker & Pommerenke, C. Cowen)

For $\varphi : \mathbb{U} \to \mathbb{U}$ *holomorphic, no fixed pt. in* $\mathbb{U}$*:*

$$\exists\, \psi \in LFT(\mathbb{U}) \;\; \& \;\; \sigma \in H(\mathbb{U}) \quad \text{such that} \quad \sigma \circ \varphi = \psi \circ \sigma$$

Moreover:

- φ *1-1 on* $\mathbb{U} \implies \sigma$ *1-1 on* $\mathbb{U}$
- $\varphi'(\omega) < 1 \implies \psi$ *hyperbolic*
- $\varphi'(\omega) = 1 \implies \psi$ *parabolic*
- *For* φ *of "parabolic type:"*

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9. Hypercyclic C_φ (φ "regular")

- φ univalent
- $\omega :=$ DW pt. on $\partial\mathbb{U}$
- φ continuous on $\overline{\mathbb{U}}$
- $\varphi(\overline{\mathbb{U}}) \subset \mathbb{U} \cup \{\omega\}$
- φ is $\in C^{(4)}$ in nbd of ω

Theorem (Bourdon & JHS) *For φ regular:*

C_φ is hypercyclic <u>unless</u> φ of parabolic non-auto. type.

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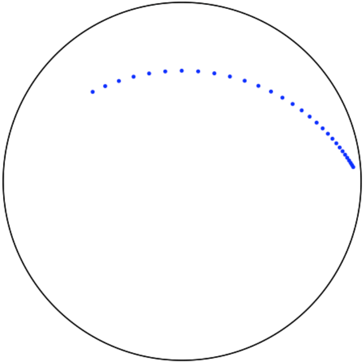

"Denjoy-Wolff" dynamics

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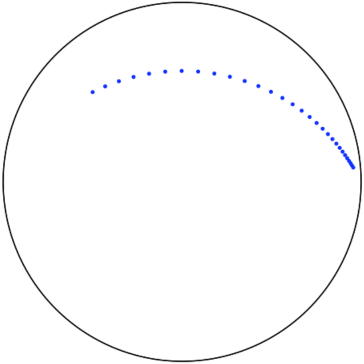

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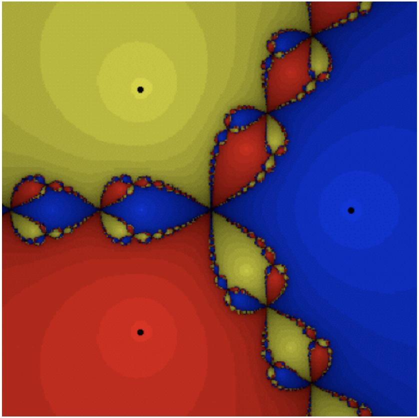

Complex dynamics

11. Complex Dynamics: The Julia Set

Definition. For $R(z)$ rational, degree > 1,

$J(R) =$ all pts. in $\hat{\mathbb{C}}$ having <u>no</u> nbd. on which $\{R_n\}$ a normal family.

Properties of $J = J(R)$

- Compact in $\hat{\mathbb{C}}$ & $\neq \emptyset$
- "Completely invariant:" $R(J) = R^{-1}(J) = J$
- $J =$ closure of backward R-orbit of any of its points.

Problem:

$$\textit{Find "natural" } \underbrace{R-\textit{invariant}}_{\mu=\mu R^{-1}} \textit{ Borel prob. measures } \mu \textit{ on } J$$

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12. Finite Blaschke products

$$\varphi(z) \;=\; \lambda \prod_{j=1}^{d} \frac{a_j - z}{1 - \overline{a_j}\, z} \qquad (|\lambda| = 1;\ a_1, a_2, \ldots a_d \in \mathbb{U})$$

Properties of φ

- Continuous on $\overline{\mathbb{U}}$, Modulus 1 on $\partial\mathbb{U}$
- $|\varphi'| \;=\; \sum_{j=1}^{d} P_{a_j} \;>\; 0$ on $\partial\mathbb{U}$
- No critical points on $\partial\mathbb{U}$, (so φ maps $\partial\mathbb{U}$ onto itself d-to-1)
- $J(\varphi) \subset \partial\mathbb{U}$: ... $= \partial\mathbb{U}$, or a Cantor set
- $J(\varphi) = \partial\mathbb{U}$ iff $\rho(\varphi_{n+1}(0), \varphi_n(0)) \to 0$

Class of examples: For $0 \le r < 1$:

$$\varphi(z) = \left(\frac{r + z}{1 + r\,z}\right)^2 \qquad J(\varphi) = \partial\mathbb{U} \;\; \textit{iff}\;\; 0 < r \le 1/3$$

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Properties of φ

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13. Invariant measures on $J = J(\varphi)$

First idea: For $\zeta \in \partial\mathbb{U}$ & $n \in \mathbb{N}$, define:

$$\mu_n^\zeta := \frac{1}{d^n} \sum_{\eta \in \varphi^{-n}\{\zeta\}} \delta_\eta \qquad (\text{so} \quad \mu_n^\zeta \varphi^{-1} = \mu_{n-1}^\zeta)$$

Theorem

$\exists\, \mu$ independent of ζ such that

$$\mu_n^\zeta \to \mu \quad \text{weak-* in } M(J)$$

Properties of μ

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14. The Transfer Operator

Definition For $f \in C(J)$, & $\zeta \in J$:

$$\mathscr{L} f(\zeta) = \frac{1}{d} \sum_{\eta \in \varphi^{-1}\{\zeta\}} f(\eta) := \int f \, d\mu_1^{\zeta}$$

Properties of $\mathscr{L}$

- $\mathscr{L}(C(J)) \subset C(J)$
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"Lyubich's Splitting Theorem" $\implies$

$$\exists \, \mu \in \text{Prob}(J) \text{ s.t. } \|\mathscr{L}^n f - \textstyle\int f \, d\mu\| \to 0 \qquad (f \in C(J)).$$

Consequence: $\forall \, \zeta \in \partial\mathbb{U}$, $\forall \, f \in C(J)$:

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15. "C_φ can 'see' the Julia set." (M. Jury 2009)

Toeplitz operator: For $f \in L^\infty(\partial\mathbb{U})$:

$$T_f g = P(fg) \qquad (g \in H^2)$$

where $P : L^2(\partial\mathbb{U}) \to H^2$ is the "Riesz Projection".

Theorem (Jury 2009)

For φ any finite Blaschke product of degree ≥ 2

$\exists\, h \in A(\mathbb{U})$ such that:

- $W \;:=\; T_h C_\varphi$ *isometric on* H^2
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Rephrase: Let $\Phi(A) \;:=\; W^* A W \quad (A : H^2 \to H^2)$

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To show: $\exists\, h \in A(\mathbb{U})$ such that if $W = T_h C_\varphi$, then

$$\Phi(T_f) := W^* T_f W = T_{\mathscr{L} f}$$

Elementary Theorem

$$A : H^2 \to H^2 \quad \textit{is Toeplitz} \quad \textit{iff} \quad T_z^* A T_z = A$$

Corollary

$C_\varphi^* T_f C_\varphi$ *is Toeplitz*

Proof:

$$\begin{aligned} T_z^*(C_\varphi^* T_f C_\varphi) T_z &= (C_\varphi T_z)^* T_f (C_\varphi T_z) \\ &= (T_\varphi C_\varphi)^* T_f (T_\varphi C_\varphi) \\ &= C_\varphi^* (T_\varphi^* T_f T_\varphi) C_\varphi \\ &= C_\varphi^* (T_{\overline{\varphi} f \varphi}) C_\varphi = C_\varphi^* T_f C_\varphi \end{aligned}$$

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So Far: $C_\varphi^* T_f C_\varphi$ *is Toeplitz.*

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Theorem (Jury 2009) $C_\varphi^* T_f C_\varphi = T_{A_\varphi f}$

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